A REMARK ON A MAXIMUM PRINCIPLE

BO YANG AND FANGYANG ZHENG

ABSTRACT. We review the maximum principle in our CAG 2013 paper, and correct some inaccuracies in the proof.

Theorem 0.1. *Let $u(x,t)$ be a smooth solution of $\frac{\partial u}{\partial t} = \Delta u + |u|^p$ with $p > 1$ on $\mathbb{R}^n \times [0,T]$ with $u(x,0) \geq 0$. Then $u(x,t) \geq 0$ for any $(x,t) \in \mathbb{R}^n \times [0,T]$*

The proof of above relies on the choice of the following cut-off function. For fixed $p > 1$, φ is a fixed smooth cut-off non-increasing function such that $\varphi = 1$ on $(-\infty, 1]$ and $\varphi = 0$ on $[2, +\infty)$. and there exists $C > 0$,

$$-C < \varphi' \leq 0, \qquad \frac{|\varphi'|}{\varphi^{\frac{3-p}{2}}} + \frac{|\varphi''|}{\varphi^{2-p}} \leq C. \tag{1}$$

Theorem 0.2 (Yang-Zheng)**.** *Let $g(t)$ be a complete solution of the Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry for $t \in [0,T]$. If the Riemannian sectional curvature of the initial metric $g(0)$ is nonnegative, so is that of $g(t)$ for any $t \in (0,T]$.*

Proof explained. Note that one can assume $A, B, C > 0$ everywhere on $\mathbb{C}^n \times [0,T]$. Suppose there is a point (z_0, t_0) where $0 < t_0 \leq T$ where the sectional curvature is negative along some real 2-plane, then $D(z_0,t_0) = AC - B^2 < 0$. By picking $r_0 > 0$ small enough we may assume that $Ric(z,t) \leq \frac{n-1}{r_0^2}$ for any $z \in B_{t_0}(z_0, r_0)$ where $B_{t_0}(z_0,r_0)$ is with respect to $g(t_0)$.

$$\begin{aligned}
& (\frac{\partial}{\partial t} - \Delta)(AC - B^2) \\
= \;& [(\frac{\partial}{\partial t} - \Delta)A]C + [(\frac{\partial}{\partial t} - \Delta)C]A - 2B[(\frac{\partial}{\partial t} - \Delta)B] \\
& -2\nabla A \cdot \nabla C + 2|\nabla B|^2 \\
= \;& A^2C + (n-2)B^2C + \frac{n}{2}C^2A + 2B^3 - 2\nabla A \cdot \nabla C + 2|\nabla B|^2.
\end{aligned} \tag{2}$$

Let φ is a fixed smooth cut-off non-increasing function such that $\varphi = 1$ on $(-\infty, 1]$ and $\varphi = 0$ on $[2, +\infty)$. Moreover,

$$-4 < \varphi' \leq 0, \qquad \frac{|\varphi''|}{\varphi^{\frac{1}{2}}} + \frac{(\varphi')^2}{\varphi^{\frac{3}{2}}} \leq 128. \tag{3}$$

Define $u(z,t) \doteqdot \varphi(\frac{d_t(z,z_0)}{ar_0})D(z,t)$, where $a > 0$ will be a sufficiently large number.

$$\begin{aligned}
& (\frac{\partial}{\partial t} - \Delta)u \\
= \;& \varphi' \frac{1}{ar_0}\Big[(\frac{\partial}{\partial t} - \Delta)d_t\Big]D + \varphi\Big[(\frac{\partial}{\partial t} - \Delta)D\Big] - 2\nabla\varphi \cdot \nabla D - \varphi'' \frac{D}{(ar_0)^2}
\end{aligned} \tag{4}$$

Denote $u_{min}(t) = min_{z \in \mathbb{C}^n} u(z,t)$, so $u_{min}(t_0) \leq u(z_0,t_0) < 0$. Assume that there exists (z_1,t_1) such that $u(z_1,t_1) = min_{t\in[0,T]} u_{min}(t) < 0$. Now we compute the right hand side of (4) at the space-time point (z_1,t_1). For simplicity, let us call it $Q(z_1,t_1)$.

First of all, Lemma 8.3 from Perelman implies:

$$(\frac{\partial}{\partial t} - \Delta)\; d_{t_1}(z,z_0) \geq -\frac{5(n-1)}{3r_0}, \tag{5}$$

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whenever $d_{t_1}(z, z_0) > r_0$.

The definition of (z_1, t_1) implies $\nabla u(z_1, t_1) = 0$. Therefore $\nabla D = -\frac{\nabla\varphi}{\varphi}D$ and $\nabla A = \frac{1}{C}(\nabla D + 2B\nabla B - A\nabla C)$.

It follows from the $F(x)$ function characterization of $U(n)$-invariant Kähler metric and a straightforward calculation that

$$\nabla_s B = \frac{2x}{v}(A - 2B), \quad \nabla_s C = \frac{2x}{v}(2B - C). \tag{6}$$

$$\begin{aligned} & Q(x_1, t_1) \\ \geq\ & \varphi\Big\{A^2C + (n-2)B^2C + \frac{n}{2}C^2A + 2B^3 - 2\nabla A \cdot \nabla C + 2|\nabla B|^2\Big\} \\ & -\frac{10(n-1)\varphi'}{3ar_0^2}D + \frac{2(\varphi')^2}{(ar_0)^2}D - \frac{\varphi'' D}{(ar_0)^2} \\ =\ & \varphi\Big[A^2C + (n-2)B^2C + \frac{n}{2}C^2A + 2B^3\Big] \\ & +\varphi\Big[-\frac{2}{C}\nabla D \cdot \nabla C - \frac{4B}{C}\nabla B \cdot \nabla C + \frac{2A}{C}|\nabla C|^2 + 2|\nabla B|^2\Big] \\ & -\frac{10(n-1)\varphi'}{3ar_0^2}D + \frac{2(\varphi')^2}{(ar_0)^2}D - \frac{\varphi'' D}{(ar_0)^2} \\ \geq\ & \varphi\Big[A^2C + (n-2)B^2C + \frac{n}{2}C^2A + 2B^3\Big] \\ & +\varphi\frac{4x^2}{v^2}\frac{2}{C}\Big[A^2C + AC^2 + 8B^3 - 6ABC\Big] \\ & -\varphi\prime\frac{1}{ar_0}\frac{2x}{Cv}|2B - C|D - \frac{10(n-1)\varphi'}{3ar_0^2}D + \frac{2(\varphi')^2}{(ar_0)^2}D - \frac{\varphi'' D}{(ar_0)^2} \end{aligned} \tag{7}$$

Note that at the point (z_1, t_1),

$$A^2C + (n-2)B^2C + \frac{n}{2}C^2A + 2B^3 \geq [B^2 - AC]^{\frac{3}{2}} = |D|^{\frac{3}{2}}, \tag{8}$$

$$A^2C + AC^2 + 8B^3 - 6ABC \geq 0. \tag{9}$$

Claim 0.3. $\frac{x|2B-C|}{Cv}$ *is uniformly bounded on* $\mathbb{C}^n \times [0, T]$.

Proof of Claim. The crucial observation is that $x - \frac{x^3}{v} = O(x^3)$ and $\frac{2x^2}{v} - 1 - \frac{1}{\sqrt{1+F'^2}} = O(x^3)$ when x small, which gives $\frac{x|2B-C|}{Cv}$ is bounded when x small.

Indeed, $v = x^2 + \frac{[F''(0)]^2}{4}x^4 + O(x^5)$ and $\sqrt{1 + (F')^2} = 1 + \frac{[F''(0)]^2}{2}x^2 + O(x^3)$, then one can check that $2\sqrt{1 + (F')^2} - \frac{v}{x^2}\sqrt{1 + (F')^2} - \frac{v}{x^2} = O(x^3)$ and $1 - \frac{x^2}{v} = \frac{[F''(0)]^2}{4}x^2 + O(x^3)$.

On the other hand, $C \geq \frac{\delta}{v}$ for x large leads to $\frac{x|2B-C|}{Cv}$ is bounded outside a compact set of $\mathbb{C}^n$. In fact,

$$\lim_{x\to+\infty} \frac{x|2B - C|}{Cv} = 0, \tag{10}$$

□

It follows from (7) that

$$\frac{d^- u_{min}(t)}{dt}\Big|_{t=t_1} \geq \frac{1}{\varphi^{\frac{1}{2}}}\Big\{|u|^{\frac{3}{2}} + [-\frac{\varphi'}{ar_0\varphi^{\frac{1}{2}}}C_1 - \frac{\varphi'}{ar_0^2\varphi^{\frac{1}{2}}}C_2 + \frac{(\varphi')^2C_3}{(ar_0)^2\varphi^{\frac{1}{2}}} + \frac{|\varphi''|}{(ar_0)^2\varphi^{\frac{1}{2}}}]u\Big\} \tag{11}$$

where C_1, C_2 and C_3 are all constants depending only on the $g(t)$ restricted to a compact subset $\mathbb{C}^n \times [0, T]$.

On the other hand, the choice of the point (z_1, x_1) implies $\frac{d^- u_{min}(t)}{dt} \leq 0$. We conclude that $\sqrt{|u(x_1,t_1)|} \leq \frac{C_5}{ar_0} + \frac{C_6}{(ar_0)^2}$. Therefore, we have

$$D(x_0,t_0) \geq u(x_1,t_1) \geq -[\frac{C_5}{ar_0} + \frac{C_6}{(ar_0)^2}]^2. \tag{12}$$

Now let a goes to infinity, we get $D(z_0,t_0) \geq 0$, which contradicts to the choice of (z_0,t_0).

□